Things you will need to know for the Real Analysis in MATH20101

Summations

Questions 13 -17 concern *Arithmetic Sums*.

13) Write the numbers $1, 2, 3, ..., n$, in a row. Write the numbers again, in a row below the first row, but in the *reverse* order. Add each of the columns. Add the resulting totals. In this way give a formula for

$$\sum_{i=1}^{n} i.$$

14) Assume that there exists a formula of the type

$$\sum_{i=0}^{n} i^3 = an^4 + bn^3 + cn^2 + dn + e, \tag{1}$$

for some real numbers $a, b, c, d$ and $e$, that will be valid for all $n \geq 0$. Substitute $n = 0, 1, 2, 3$ and 4 to find five different equations satisfied by $a, b, c, d$ and $e$. Solve and thus give a formula for

$$\sum_{i=0}^{n} i^3$$

valid for all $n \geq 0$.

This could take you some time! In fact, you should end up with only *four* equations in four unknowns. You can write this system as a matrix equation, the question then becomes one of inverting a $4 \times 4$ matrix. This is an excellent problem on which to apply Gaussian elimination as learnt in First year Linear Algebra.

15) Prove by induction that

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

for all $n \geq 1$.

**Note**, though you will never have need of them there are results for $\sum_{i=1}^{n} i^\ell$ for every $\ell \in \mathbb{N}$, and the next two such results for $\ell = 5$ and 6 are

$$\sum_{i=1}^{n} i^5 = \frac{n^2 (n+1)^2 (2n^2+2n-1)}{12}$$
$$= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2.$$

and

$$\sum_{i=1}^{n} i^6 = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$$
$$= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n.$$

**Aside** What is interesting here is that, for all $\ell \in \mathbb{N}$ the sums of $\ell$-th powers of integers, $\sum_{i=1}^{n} i^\ell$, **are** integers. Therefore the fractions

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}, \frac{n^2(2n^2+2n-1)(n+1)^2}{12}$$
$$\text{and } \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$$

are integers for all $n \geq 1$. Alternatively, the polynomials

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}, \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

and

$$\frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n,$$

do **not** have integer coefficients but are, nonetheless, integers for every $n \geq 1$. I would claim this is not obvious at first sight.

**Question** Use the Theory of Congruences to show that

$$n(n+1)(2n+1)\left(3n^2+3n-1\right) \equiv 0 \bmod 30,$$
$$n^2\left(2n^2+2n-1\right)(n+1)^2 \equiv 0 \bmod 12,$$
$$n(n+1)(2n+1)\left(3n^4+6n^3-3n+1\right) \equiv 0 \bmod 42,$$

for all $n \geq 1$

16) (2003) Evaluate

$$\sum_{i=1}^{n}\left(1+2\frac{i}{n}\right)^2 \quad \text{and} \quad \sum_{i=1}^{n}\left(1+2\frac{i-1}{n}\right)^2.$$

13) (2004) Evaluate

$$\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^3 \quad \text{and} \quad \sum_{i=1}^{n}\left(\frac{i}{n}\right)^3.$$

## Questions 18 -21 concern Geometric Sums.

18) Let $x \in \mathbb{R}$ and set

$$S = x + x^2 + x^3 + ... + x^n.$$

Look at $xS$, rewrite in terms of $S$, rearrange and thus give a formula for

$$\sum_{i=1}^{n} x^i,$$

valid for $x \neq 1$.

Check your answer using Question 1 iii) above.

19) Give a formula for

$$\sum_{i=1}^{n} \frac{1}{x^i},$$

valid for $x \neq 0, 1$.

20) (2006) Evaluate

$$\sum_{i=1}^{n} 2\left(2\eta^i\right)^2\left(2\eta^i - 2\eta^{i-1}\right) \quad \text{and} \quad \sum_{i=1}^{n} 2\left(2\eta^{i-1}\right)^2\left(2\eta^i - 2\eta^{i-1}\right).$$

where $\eta$ is the $n$-th root of a positive $X$, i.e. $\eta^n = X$.

21) (2009) Evaluate

$$\sum_{i=1}^{n}\left(\frac{1}{\eta^{i-1}}\right)^2\left(\eta^i - \eta^{i-1}\right) \quad \text{and} \quad \sum_{i=1}^{n}\left(\frac{1}{\eta^i}\right)^2\left(\eta^i - \eta^{i-1}\right)$$

where $\eta$ is the $n$-th root of a positive $X$, i.e. $\eta^n = X$.